Limiting behaviour of the Ricci flow

Natasa Sesum

Abstract

We will consider a τ-*flow*, given by the equation $\frac{d}{dt}g_{ij} = -2R_{ij} + \frac{1}{\tau}g_{ij}$ on a closed manifold M, for all times $t \in [0, \infty)$. We will prove that if the curvature operator and the diameter of $(M, g(t))$ are uniformly bounded along the flow, then we have a sequential convergence of the flow toward the solitons.

1 Introduction

The studies of singularities and the limiting behaviours of solutions of various geometric partial differential equations have been important in geometric analysis. One of these important geometric equations is so called Ricci flow equation, itroduced by Richard Hamilton in [6]. It is the equation $\frac{d}{dt}g_{ij}(t) = -2R_{ij}$, for a Riemannian metric $g_{ij}(t)$. The short time existence of this equation was proved by Hamilton in [6] and somewhat later the proof was significantly simplified by DeTurck in [4]. Hamilton showed that the Ricci flow preserves the positivity of the Ricci tensor in dimension three and of the curvature operator in all dimensions. This observation helped him to prove the convergence results in dimensions three and four, towards metrics of constant positive curvatures (in the case of positive Ricci curvature and positive curvature operator respectively).

Besides the short time existence we can also study a long time existence of the Ricci flow. There is a well known Hamilton's result.

Theorem 1 (Hamilton). *For any smooth initial metric on a compact manifold there exists a maximal time T on which there is a unique smooth solution to the ricci flow for $0 \le t < T$. Either $T = \infty$ or else the curvature is unbounded as $t \to T$.*

One can ask what happens to a solution if it exists for all times and under which conditions it will converge to a metric that will have nice properties. In the case of dimension three with positive Ricci curvature and dimension four with positive curvature operator we know that a solution converges to an Einstein metric. In general, we can not expect to get an Einstein metric in the limit. We can expect to get a solution to an evolution equation which moves under a one-parameter subgroup of the symmetry group of the equation. These kinds of solutions are called solitons.

Our goal in this paper is to prove the following theorem.

Theorem 2 (Main Theorem). *Consider the flow*

$$\frac{dg_{ij}}{dt} = -2R_{ij} + \frac{1}{\tau}g_{ij} \tag{1}$$

on a compact manifold M, where $\tau > 0$ is fixed, $|Rm| \le C$ and $diam(M, g(t) \le C$ $\forall t \in [0, \infty)$. Then for every sequence of times $t_i \to \infty$ there exists a subsequence, so that $g(t_i + t) \to h(t)$ and $h(t)$ is a Ricci soliton.

The organization of the paper is as follows. In section 3 we will prove some properties of $\mu(g, \tau)$ that has been introduced by Perelman in [10]. They will be useful in the later sections of the paper. In section 3 we will prove Theorem 2.

Acknowledgements: The author would like to thank her advisor Gang Tian for bringing this problem to her attention and for constant help and support.

2 Preliminaries

Perelman's functional $\mathcal{W}$ and its properties will play an important role in the proof of Theorem 2. M will always denote a compact manifold, and $(g_{ij})_t = -2R_{ij} + \frac{1}{\tau}g_{ij}$ will be a flow that we will be considering throughout the whole paper. Perelman's functional $\mathcal{W}$ has been introduced in [10].

$$W(g,f,\tau) = (4\pi\tau)^{-\frac{n}{2}}\int_M e^{-f}[\tau(|\nabla f|^2 + R) + f - n]dV_g.$$

We will consider this functional restricted to f satisfying

$$\int_M (4\pi\tau)^{-\frac{n}{2}}e^{-f}dV = 1. \tag{2}$$

$\mathcal{W}$ is invariant under simultaneous scaling of τ and g. Perelman showed that the Ricci flow can be viewed as a gradient flow of functional $\mathcal{W}$. Let $\mu(g,\tau) = \inf \mathcal{W}(g,f,\tau)$ over smooth f satisfying (2). It has been showed by Perelman that there always exists a smooth minimizer on a closed manifold M, that $\mu(g,\tau)$ is negative for small $\tau > 0$ and that it tends to zero as $\tau \to 0$. One of the most important properties of $\mathcal{W}$ is the monotonicity formula.

Theorem 3 (Perelman). $\frac{d}{dt}\mathcal{W} = \int_M 2\tau|R_{ij}+\nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}|^2(4\pi\tau)^{-\frac{n}{2}}e^{-f}dV \geq 0$ *and therefore $\mathcal{W}$ is increasing along the Ricci flow.*

One of the very important applications of the monotonicity formula is noncollapsing theorem for the Ricci flow that has been proved by Perelman in [10].

Definition 4. Let $g_{ij}(t)$ be a smooth solution to the Ricci flow $(g_{ij})_t = -2R_{ij}(t)$ on $[0,T)$. We say that $g_{ij}(t)$ is loacally collapsing at T, if there is a sequence of times $t_k \to T$ and a sequence of metric balls $B_k = B(p_k, r_k)$ at times t_k, such that $\frac{r_k 2}{t_k}$ is bounded, $|\mathrm{Rm}|(g_{ij}(t_k)) \leq r_k^{-2}$ in B_k and $r_k^{-n}\mathrm{Vol}(B_k) \to 0$.

Theorem 5. *If M is closed and $T < \infty$, then $g_{ij}(t)$ is not locally collapsing at T.*

3 Sequential convergence of a τ-flow

Definition 6. *τ-flow* is given by the equation

$$\frac{d}{dt}g_{ij} = -2R_{ij} + \frac{1}{\tau}g_{ij}, \tag{3}$$

for $\tau > 0$.

We want to prove the Theorem 2 in this section.

3.1 Convergence toward the solutions of the Ricci flow

In order to prove Theorem 2 we will first show that it is reasonable to expect a convergence toward a smooth manifold, i.e. that a limit manifold will not collapse.

Claim 7. *Consider the flow as above. For every fixed $\tau > 0$ there exists a constant C such that $\mathrm{Vol}_{g(t)}(M) \geq C$ for every t, i.e. we have a uniform lower bound on the volumes.*

Proof. Assume that the claim is not true, i.e. that there exists a sequence t_i s.t. $\mathrm{Vol}_{g(t_i)}(M) \to 0$ as $i \to \infty$. Let $\bar{g}(s) = c(s)g(t(s))$ be unnormalized flow, for $s \in [0, \tau)$, where:

$$t(s) = -\tau \ln(1 - \frac{s}{\tau}).$$

$$c(s) = 1 - \frac{s}{\tau}.$$

$$R(\bar{g}) = \frac{R(g)}{c(s)}.$$

Find s_i, such that $t(s_i) = t_i$. We get that $s_i = \tau(1 - e^{-\frac{t_i}{\tau}})$. $s_i \to \tau$ as $i \to \infty$. Let

$$\max_{M\times[0,s_i]} |Rm|(\bar{g}(s)) = Q_i, \tag{4}$$

and assume that the maximum is achieved at p_i. By the corollary of Perelman's noncollapsing theorem we have that:

$$\frac{\mathrm{Vol}_{\bar{g}(t)} B(p_i, r)}{r^n} \geq C_1,$$

for $r \leq C\sqrt{\frac{\tau}{Q_i}}$ and $t \in [0, s_i)$. Choose $r = C\sqrt{\frac{\tau}{Q_i}}$ and $t = s_i$.

$$(\sqrt{Q_i})^n \mathrm{Vol}_{\bar{g}(s_i)} B(p_i, C\sqrt{\frac{\tau}{Q_i}}) \geq (C\sqrt{\tau})^n C_1 = \tilde{C}.$$

Since $\mathrm{Vol}_{\bar{g}(s_i)} B(p_i, r) = c(s_i)^{\frac{n}{2}} \mathrm{Vol}_{g(t_i)} B(p_i, \tilde{r})$, where $\tilde{r}$ might be a different radius as a matter of scaling and since $Q_i \leq \frac{C}{c(s_i)}$ (because the curvature of $g(t)$ is uniformly bounded), we get that:

$$\mathrm{Vol}_{g(t_i)}(M) \geq \tilde{C}/C,$$

where $\tilde{C}$ and C do not depend on i. Let $i \to \infty$ in the previous inequality to get a contradiction. Therefore we have a uniform lower bound on volumes.

$\square$

Remark 8. *The assumptions of the Theorem 2 and the result of Claim 7 imply the uniform bounds on the curvature tensors, uniform upper bound on the diameters and uniform lower bounds on the volumes. Similarly like in the case of unnormalized flow, uniform bounds on the curvatures gives us uniform bounds on all covariant derivatives, so by Hamilton's compactness theorem, for every sequence $t_i \nearrow \infty$ as $i \to \infty$, there exists a subsequence (call it again t_i), such that $(M, g(t_i + t))$ converges to $(M, h(t))$, in the sense that there exist diffeomorphisms $\phi_i : M \to M$, so that $\phi_i^* g(t_i + t)$ converge uniformly together with their covariant derivatives to metrics $h(t)$ on compact subsets of $M \times [0, \infty)$. Moreover, $h(t)$ is a solution of a τ-flow as well.*

3.2 Continuity of the minimizers for $\mathcal{W}$

We will recall a definition of Perelman's functional $\mathcal{W} = (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-f}[\tau(R + |\nabla f|^2) + f - n]dV$. The constraint on f for this functional is (*) $(4\pi\tau)^{-\frac{n}{2}} \int e^{-f} dV =$

1. Let $\mu(g,\tau) = \inf \mathcal{W}(g,f,\tau)$ under the constraint (*). This infinimum has been achieved by some smooth minimizer f. Perelman has also proved that for a fixed metric g, $\lim_{\tau\to 0}\mu(g,\tau) = 0$ and $\mu(g,\tau) < 0$ for a small value of $\tau > 0$.

In the case of a τ-flow $g(t)$, $\tau > 0$ is being fixed in time, and by the monotonicity formula for $\mathcal{W}$ we have that $\mu(g(t),\tau)$ is increasing along the flow. Therefore, there exists $\lim_{t\to\infty}\mu(g(t),\tau)$.

Claim 9. $\lim_{t\to\infty}\mu(g(t),\tau)$ *is finite.*

Proof. Assume that $\lim_{t\to\infty}\mu(g(t),\tau) = \infty$. Then, $\forall i$, $\exists t_i$ s.t. $\mu(g(t_i),\tau) \geq i$. There exists a subsequence (call it t_i) such that (M, g_i) converges to (M, h), for some metric h. From the first part of Lemma 10 we get that $\mu(g(t_i),\tau) < \mu(h,\tau) + \epsilon$, for i big enough. Letting $i \to \infty$ we get a contradiction. $\square$

Lemma 10. *If (M, g_i) tend to (M, h) when $i \to \infty$, where $g_i = g(t_i)$ and $t_i \nearrow \infty$, then* $\lim_{i\to\infty}\mu(g_i,\tau) = \mu(h,\tau)$.

Proof.

$$\mu(h,\tau) = \int_M (\tau(|\nabla f|^2 + R(h)) + f - n)(4\pi\tau)^{-\frac{n}{2}} dV_h.$$

Since $\phi_i^* g_i \to \infty$ uniformly with their covariant derivatives, if $\epsilon > 0$ is fixed, there exists some big i_0, so that for $i \geq i_0$

$$\mu(h,\tau) \geq \int_M (\tau(|\nabla f|^2 + R(\tilde{g}_i)) + f - n)(4\pi\tau)^{-\frac{n}{2}} dV_{\tilde{g}_i} - \frac{\epsilon}{2},$$

where $\tilde{g}_i = \phi^* g_i$. Change the variables in the above integral by diffeomorphism ϕ_i.

$$\mu(h,\tau) \geq \int_M (\tau(|\nabla_i f_i|^2 + R(g_i)) + f_i - n)(4\pi\tau)^{-\frac{n}{2}} dV_{g_i} - \frac{\epsilon}{2},$$

where $f_i = \phi^* f$. Perturb a little bit f_i to get $\tilde{f}_i$, by a quantity that tends to zero, so that $\int_M e^{-\tilde{f}_i}(4\pi\tau)^{-\frac{n}{2}} dV_{g_i} = 1$. Since our geometries are uniformly bounded, for big enough i_0 we will have

$$\mu(h,\tau) \geq \mathcal{W}(g_i,\tilde{f}_i,\tau) - \epsilon \geq \mu(g_i,\tau) - \epsilon. \tag{5}$$

Let $u_i = e^{-\frac{f_i}{2}}$. We have seen that minimizing $\mu(g_i, \tau)$ by f_i is equivalent to minimizing the following expression in u_i:

$$\int_M \tau(4|\nabla_i u_i|^2 + R_i u_i^2) - 2u_i^2 \ln u_i - n u_i^2)(4\pi\tau)^{-\frac{n}{2}} dV_{g_i}.$$

The minimizer u_i has to satisfy the following elliptic differential equation

$$\tau(-4\Delta_i u_i + R_i u_i) - 2u_i \ln u_i - n u_i = \mu_{i,\tau} u_i. \tag{6}$$

$\mu_{i,\tau}$ is uniformly bounded, since there is a finite $\lim_{t\to\infty} \mu(g(t), \tau)$. Now we can easily get:

$$\int_M u_i^2 (4\pi\tau)^{-\frac{n}{2}} dV_i \leq C, \tag{7}$$

$$\tau \int_M |\nabla_i u_i|^2 (4\pi\tau)^{-\frac{n}{2}} dV_i \leq C, \tag{8}$$

i.e. $u_i \in W^{1,2}$ with

$$||u_i||_{W^{1,2}} \leq C \quad \forall i.$$

From (6), by standard regularity theory of partial differential equations and Sobolev embedding theorems, we get that $u_i \in W^{k,p}$ with uniformly bounded $W^{k,p}$ norms, where $p < \frac{2n}{n-2}$, and therefore with uniformly bounded $C^{2,\alpha}$ norms, i.e. $||u_i||_{C^{2,\alpha}} \leq C$. Furthermore,

$$\begin{aligned} \mu(g_i, \tau) &= \int_M (\tau(4|\nabla_i u_i|^2 + R_i u_i^2) - 2u_i^2 \ln u_i - n u_i^2)(4\pi\tau)^{-\frac{n}{2}} dV_i \\ &= \int_M \tau(|\tilde{\nabla}\tilde{u_i}|^2 4 + \tilde{R}_i \tilde{u_i}^2) - 2\tilde{u_i} \ln \tilde{u_i} - n\tilde{u_i}^2)(4\pi\tau)^{-\frac{n}{2}} dV_{\tilde{g_i}}, \end{aligned} \tag{9}$$

where $\tilde{u_i} = \phi_i^* u_i$. $\phi_i^* g_i$ is close to h and therefore for i big enough, ϕ_i is almost an isometry, so $D_j \phi_i^{-1}$ can be uniformly bounded in terms of bounds on g_i and h, g_i can be bounded in terms of h. We cover M with finitely many geodesic balls of fixed radius ρ (we can do it since we have a uniform bound on the injectivity radii from below). We use local coordinates in each of the balls to get:

$$|\tilde{\nabla}_i \tilde{u}_i|^2 = \tilde{g}_i^{jk} D_j(u_i \circ \phi_i^{-1}) D_k(u_i \circ \phi_i^{-1}).$$

$$|\tilde{\nabla}\tilde{u}_i|^2 = \tilde{g}_i^{jk}(D_j u_i)(D_k u_i)(\phi_i^{-1})D_j\phi_i^{-1}D_k\phi_i^{-1}.$$

Now we can easily conclude that we have a uniform bound on $|\tilde{\nabla}u_i|^2$. Since the integrand in (9) is uniformly bounded in i, and since $\tilde{g}_i$ uniformly converge with their covariant derivatives to h, we have that for i large enough

$$\mu(g_i,\tau) \geq \int_M (\tau(4|\nabla_h\tilde{u}_i|^2 + R_h\tilde{u}_i^2) - 2\tilde{u}_i\ln\tilde{u}_i - n\tilde{u}_i^2)(4\pi\tau)^{-\frac{n}{2}}dV_h - \epsilon.$$

Since $l_i = \int_M \tilde{u}_i^2(4\pi\tau)^{-\frac{n}{2}}dV_h$ is close to 1 when $i \to \infty$, taking $\bar{u}_i = \frac{\tilde{u}_i}{l_i}$ and using all the uniform bounds that we have got by now

$$\mu(g_i,\tau) \geq \mathcal{W}(h,\bar{u}_i,\tau) - \epsilon \geq \mu(h,\tau) - \epsilon.$$

By the previous inequality (for i big enough) and by (5) we get $\lim_{i\to\infty}\mu(g_i,\tau) = \mu(h,\tau)$. □

Following the notation from the previous lemma, by Arzela-Ascoli theorem there exists a subsequence, u_i, so that it converges in $C^{2,\alpha}$ norm to some function u. We can also get the higher order uniform estimates on u_i in a similar manner as in Lemma 10. Therefore, to show that a sequence of minimizers for $\mu(g_i,\tau)$ converges to a minimizer of $\mu(h,\tau)$ it is enough to show the following lemma.

Lemma 11. *$\exists C > 0$ so that $u_i \geq C > 0$ $\forall i$ and $\forall x \in M$*

Proof. Assume that there exists a sequence u_i and $p_i \in M$, such that $0 < u_i(p_i) < \frac{1}{2i}$. M is compact and therefore there is a subsequence, $\{p_i\}$ converging to $p \in M$ when $i \to \infty$. $C^{2,\alpha}$ norms of u_i are uniformly bounded in i and therefore $u_i(p) < u_i(p_i) + C\mathrm{dist}_{g_i}(p,p_i) \to 0$ as $i \to \infty$. Let u be a limit of $\{u_i\}$ in $C^{2,\alpha}$ norm. Then $u(p) = 0$. Take a geodesic ball $B(p,r)$. Let $f \in C_0^\infty(M)$ be a C^∞ function of r alone, compactly supported in $B(p,r)\backslash\{p\}$.

$$\int_M (\tau(\nabla u_i\nabla f + R_i u_i f) - 2u_i f\ln u_i - nu_i f - \mu(g_i,\tau)u_i f)dV_i = 0.$$

For this f, letting $i \to \infty$, using the result of the previous lemma and the fact that the integrand in the previous integral is uniformly bounded in i we get

$$\int_M (\tau(\nabla u \nabla f + fuR(h)) - 2uf \ln u - nuf - \mu(h,\tau) fu) dV_h = 0.$$

Proceeding in the same manner as in [11] we can get that $u \equiv 0$ in some small ball around p. Using the connectedness argument, $u \equiv 0$ in M. On the other hand $\int_M u_i^2 (4\pi\tau)^{-\frac{n}{2}} dV_i = 1$ and letting $i \to \infty$ we get a contradiction. $\square$

If we write down the equations (6) for all $\{u_i\}$, letting $i \to \infty$, keeping in mind the previous lemma we get

$$\tau(-4\Delta u + R(h))u - 2u \ln u - nu = \mu(h,\tau)u,$$

i.e. u is the minimizer for $\mu(h,\tau)$.

So far we have proved the following theorem

Theorem 12. *If $(M, g_i) \to (M, h)$ as $i \to \infty$, then for a given $\tau > 0$, if $\mu(g_i, \tau) = \mathcal{W}(g_i, f_i, \tau)$, then $f_i \to f$ in $C^{2,\alpha}$ norm, where $\mu(h,\tau) = \mathcal{W}(h, f, \tau)$.*

3.3 Further estimates on the minimizers

In this subsection we want to use the minimizers f_t for $\mathcal{W}$ at different times to construct the functions $f_t(s)$ for $s \in [0,t]$. By using the parabolic regularity we will be able to get the uniform estimates on $C^{k,\alpha}$ norms of $f_t(s)$. This will enable us to take a limit of this functions along the sequences. This limits are the functions that will turn out to be the potential functions that come into the equations describing the soliton type solutions arising in a limit.

For any t we can find f_t such that $\mathcal{W}(g(t), f_t, \tau) = \mu(g(t), \tau)$. If we flow f_t backward, we will get functions $f_t(s)$ that satisfy

$$\begin{aligned} \frac{df_t(s)}{ds} &= -R(s) - \Delta f_t(s) + |\nabla f_t(s)|^2 + \frac{n}{2\tau}, \\ f_t(t) &= f_t. \end{aligned}$$

We know that minimizing $\mathcal{W}$ in f is equivalent to minimizing the corresponding functional in $\tilde{u}$, where $\tilde{u}_t = e^{-\frac{f_t}{2}}$. Let $u_t(s) = \tilde{u}_t^2(s)$. The equation for $u_t(s)$ is

$$\begin{aligned}
\frac{du_t}{ds} &= -\Delta u_t + (-\frac{n}{2\tau} + R(s))u_t(s), \\
u_t(t) &= u_t.
\end{aligned}$$

By the monotonicity of $\mathcal{W}$ along the flow (1) we have that

$$\mu(g(s),\tau) \le \mathcal{W}(g(s), f_t(s), \tau) \le \mathcal{W}(g(t), f_t, \tau) = \mu(g(t), \tau).$$

First of all, there exists $\lim_{t\to\infty} \mu(g(t),\tau)$. It is finite, since for every sequence $t_i \to \infty$ there exists a subsequence such that $g(t_i) \to h(0)$ and by Lemma 10 from the previous section, we have that $\mu(g(t_i),\tau) \to \mu(h(0),\tau)$.

Instead of functional $\mathcal{W}(g(s), f_t(s), \tau)$ we can consider the equivalent functional which depends on $\tilde{u}_t(s) = e^{-f_t(s)/2}$.

$$\mathcal{W}(u_t(s)) = \int_M [\tau(4|\nabla\tilde{u}_t(s)|^2 + R\tilde{u}_t(s)^2) - \tilde{u}_t(s)^2 \log\tilde{u}_t(s)^2 - n\tilde{u}_t(s)^2](4\pi\tau)^{-n/2} dV, \tag{10}$$

where $\tilde{u}_t$ satisfy

$$\tau(-4\Delta\tilde{u}_t + R\tilde{u}_t) - 2\tilde{u}_t \ln\tilde{u}_t - n\tilde{u}_t = \mu(g(t),\tau)\tilde{u}_t,$$

since f_t is a minimizer for $\mathcal{W}$. Since $\mu(g(t),\tau)$ is uniformly bounded, as in the previous section we can get that $C^{2,\alpha}$ norms of $\tilde{u}_t$ are uniformly bounded. This implies that $C^{2,\alpha}$ norms of u_t are uniformly bounded. Before we proceed with further discussion notice the following.

Remark 13. $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f_t(s)} dV_{g(s)} = 1$. *This is a simple consequence of the fact that* $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f_t} dV_{g(t)} = 1$, *since* f_t *is a minimizer for* $\mathcal{W}$ *with respect to* $g(t)$, *and the following backward parabolic equation*

$$\frac{d}{ds} f_t(s) = -\Delta f_t(s) + |\nabla f_t(s)|^2 - R + \frac{n}{2\tau}.$$

Namely,

$$
\begin{aligned}
\frac{d}{ds}\Big(\int_M e^{-f_t(s)} dV_{g(s)}\Big) &= \int_M e^{-f_t(s)}\Big(\Delta f_t(s) - |\nabla f_t(s)|^2 + R - \frac{n}{2\tau} - R + \frac{n}{2\tau}\Big) dV_{g(s)} \\
&= \int_M \Delta(e^{-f_t(s)} dV_{g(s)}) = 0
\end{aligned}
$$

Since log is a concave function and $\tilde{u}_t(s)^2(4\pi\tau)^{-n/2}dV$ is a probability measure, we have by Jensen and Sobolev inequalities

$$
\begin{aligned}
\int_M \tilde{u}_t(s)^2 \log \tilde{u}_t(s)^2 (4\pi\tau)^{-n/2} dV &= \frac{n-2}{2}\int_M \tilde{u}_t(s)^2 \log \tilde{u}_t(s)^{4/(n-2)} (4\pi\tau)^{-n/2} dV \\
&\le \frac{n-2}{2}\log \int_M \tilde{u}_t(s)^{2n/(n-2)}(4\pi\tau)^{-n/2} dV \\
&\le \frac{n-2}{2}\log\Big[C\int_M (|\nabla \tilde{u}_t(s)|^2 + \tilde{u}_t(s)^2) dV\Big]^{(n-2)/n} + \\
&+ \frac{n-2}{2}\log(4\pi\tau)^{-n/2} \\
&= \frac{n}{2}\log C \int_M \tau(|\nabla \tilde{u}_t(s)|^2 + \tilde{u}_t(s)^2)(4\pi\tau)^{-n/2} dV.
\end{aligned}
$$

This inequality shows that

$$
\tau \int_M |\nabla \tilde{u}_t(s)|^2 (4\pi\tau)^{-n/2} dV \le C. \tag{11}
$$

The constant C does not depend either on t or $s \in [0, t]$. To conclude, we have the following estimates

$$
\int_M |\tilde{u}_t(s)|^2 (4\pi\tau)^{-\frac{n}{2}} dV_s \le C_1
$$

$$
\tau(4\pi\tau)^{-\frac{n}{2}} \int_M |\nabla_s \tilde{u}_t(s)|^2 dV_s \le C_2,
$$

that is we have that $|\tilde{u}_t|_{W_{1,2}} \le C$ for a uniform constant C.

Take a sequence $t_i \to \infty$. There exists a subsequence such that $g(t_i + t) \to h(t)$ when $i \to \infty$, where $h(t)$ is a Ricci flow on M. This follows from Hamilton's compactness theorem ([7]). Fix $A > 0$. f_t will be a minimizer for $\mathcal{W}$ with respect to $g(t)$, which we flow backward, for every t. Let $s \in [0, A]$.

Lemma 14. *For every $A > 0$ there exists $\delta = \delta(A) > 0$ such that $u_{t+A}(t + s) \geq \delta > 0$ for all t and all $s \in [0, A]$.*

Proof. Assume that the statement of the lemma is not true. In that case there would exist a sequence s_i such that $\min_M u_{s_i+A}(s_i+a_i) \to 0$ as $i \to \infty$, for some $a_i \in [0, A]$. Consider the equation

$$\begin{aligned}
\frac{d}{dt}u_{s_i+A}(s_i+t) &= -\Delta u_{s_i+A}(s_i+t) + (R - \frac{n}{2\tau})u_{s_i+A}(s_i+t), \\
u_{s_i+A}(s_i+A) &= u_{s_i+A},
\end{aligned}$$

for $t \in [0, A]$. Let $\hat{u}_i(s_i + t) = \min_M u_{s_i+A}(s_i + t)$. Then $\Delta\hat{u}_{s_i+A}(s_i + t) \geq 0$ and

$$\frac{d}{dt}\hat{u}_i(s_i+t) \leq C\hat{u}_i(s_i+t),$$

where C is a uniform constant. If we integrate it with respect to t, we get

$$\hat{u}_i(s_i + A) \leq e^{CA}\hat{u}_i(s_i + t).$$

Since $\hat{u}_i(s_i + A) = \min_M u_{s_i+A}$ and since by Lemma 11 we know that there exists a constant δ such that $u_{s_i+A} \geq \delta > 0$, we have that $u_{s_i+A}(s_i + t) \geq \delta(A) > 0$ for all i and all $t \in [0, A]$. This contradicts our assumption that $\hat{u}_i(s_i + a_i) \to 0$ as $i \to \infty$. □

Lemma 15. *For every $A > 0$ there exists $C(A)$ such that*

1. $\int_M u_t(s)^2 dV_{g(s)} \leq C(A)$.
2. $\int_M |\nabla u_t(s)|^2 dV_{g(s)} \leq C(A)$,

for all $t \geq A$, $s \in [t - A, t]$.

Proof. We will consider the equation

$$\begin{aligned}
\frac{d}{ds}u_t(s) &= -\Delta u_t(s) + (R - \frac{n}{2\tau})u_t(s) \\
u_t(t) &= u_t,
\end{aligned}$$

where $u_t = e^{-f_t}$ and f_t is a minimizer for $\mathcal{W}$ with respect to metric $g(t)$. Let $\hat{u}_t(s) = \max_M u_t(s)$. Then

$$\frac{d}{ds}\hat{u}_t(s) \geq -C\hat{u}_t(s),$$

where $C > 0$ is a uniform constant that does not depend either on s or t, but on the uniform bounds on geometries $g(t)$. If we integrate it with respect to s we get

$$\hat{u}_t = \hat{u}_t(t) \geq e^{-CA}\hat{u}_t(s),$$

for any $s \in [t-A, t]$. On the other hand, we have already proved in the previous section that $C^{2,\alpha}$ norms of u_t are uniformly bounded in $t \in [0,\infty)$. Therefore we get that $0 \leq u_t(s) \leq C(A)$ on M for all $t \in [A,\infty)$ and all $s \in [t-A, t]$. Now we immediately get part 1 of our claim. For part 2 notice that

$$\int_M |\nabla u_t(s)|^2 dV_{g(s)} = 4\int_M u_t(s)|\nabla\tilde{u}_t(s)|^2 dV_{g(s)} \leq \tilde{C}(A),$$

since $\int_M |\nabla\tilde{u}_t(s)|^2$ is uniformly bounded for all $t \geq A$ and $s \in [t-A, t]$. $\square$

The previous two lemmas tell us that in order to find the uniform estimates on $f_{t_i+A}(t_i+s)$ for $s \in [0, A]$, it is enough to find the uniform $C^{k,\alpha}$ estimates on $u_{t_i+A}(t_i+s)$. Our main goal in this section is to prove the following theorem.

Theorem 16. *Under the assumptions of the main theorem, with the notations as above, for every $A > 0$ there exists a uniform constant C, depending on A such that $|u_t(s)|_{C^{2,\alpha}} \leq C$ for all $t \geq A$, $\forall s \in [t-A, t]$.*

Proof. Consider the equation

$$\frac{d}{ds}u_t(s) = -\Delta u_t(s) + (R(s) - \frac{n}{2\tau})u_t(s),$$

for $t \in [A,\infty)$ and $s \in [t-A, t]$. All our further estimates will depend on A. We will use C to denote different absolute constants that depend on A

and the uniform bounds on our geometries $g(t)$. Denote by $h = h_t(s) = (-\frac{n}{2\tau} + R(s))u_t(s)$. Omit the subscript t.

$$\frac{d}{ds}u + \Delta u = h.$$

$$\int_M h^2 = \int_M (\frac{d}{ds}u)^2 + 2\int_M \frac{d}{ds}u\Delta u + \int_M (\Delta u)^2, \tag{12}$$

where we should keep in mind that the metric depends on s.

$$\begin{aligned}
\int_M \frac{d}{ds}u\Delta u &= -\int_M g^{ij}\nabla_i(\frac{d}{ds}u)\nabla_j u dV_s \qquad (13)\\
&= -\frac{1}{2}\frac{d}{ds}\int_M |\nabla u|^2 dV_s - \int_M |\nabla u|^2(\frac{n}{2\tau} - R)dV_s + \\
&+ \int_M g^{pi}g^{qj}D_i u D_j u(2R_{pq} - \frac{1}{2\tau}g_{pq})dV_s,
\end{aligned}$$

where the second term on the right hand side of (13) comes from taking the derivative of the volume element and the third term appears from taking the derivative of g^{ij}. Denote the former one by J_1 and the latter one by J_2.

$$\begin{aligned}
\int_M (\Delta u)^2 &= \int_M g^{ij}D_iD_j u g^{kl}D_kD_l u\\
&= -\int_M g^{ij}g^{kl}D_j u D_i D_k D_l u\\
&= -\int_M g^{ij}g^{kl}D_j u D_k D_i D_l u + \int_M g^{ij}g^{kl}D_j u R^l_{iks}D_s u\\
&= I + \int_M g^{ij}g^{kl}D_k D_j u D_i D_l u\\
&= I + \int_M |\nabla^2 u|^2,
\end{aligned}$$

where $I = \int_M g^{ij}g^{kl}D_j u R^l_{iks}D_s u$. Let $l \in (t-A, t)$ where $A > 0$. Integrating the equation (12) in s, from l to t gives

$$\begin{aligned}
&\int_l^t (\int_M (\frac{d}{ds}u)^2 dV_s)ds + \int_M |\nabla u|^2 dV_s|_{s=l} + \int_l^t \int_M |\nabla^2 u|^2 dV_s ds\\
&\quad = \int_l^t \int_M h^2 + \int_M |\nabla u|^2 dV_s|_{s=t} + \int_l^t (2J_1 + 2J_2 + I).
\end{aligned}$$

$$\int_l^t J_1 \leq AC \sup_{s\in(t-A,t)} \int_M |\nabla u|^2 dV_s \leq \tilde{C},$$

for every t. Similarly we get estimates for J_2 and I. From all these estimates we can conclude the following

$$\int_{t-A}^t \int_M (\frac{d}{ds} u_t(s))^2 dV_s ds \leq C. \tag{14}$$

$$\int_{t-A}^t \int_M |\nabla^2 u_t(s)|^2 dV_s ds \leq C. \tag{15}$$

$$\sup_{s\in(t-A,t)} \int_M |\nabla u|^2 dV_s \leq C, \tag{16}$$

where $C = C(A)$. Let $\tilde{u}_t = \frac{d}{ds} u_t(s)$ (we will not confuse this $\tilde{u}_t$ with one defined at the beginning of this section). Omit the subscript t.

$$\frac{d}{ds}\tilde{u} = -D_s \Delta_s u + \frac{d}{ds}[(R - \frac{n}{2\tau})u].$$

Multiply the equation by $\tilde{u}$ and integrate it along M.

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds}\int_M |\frac{d}{ds}u|^2 dV_s &= -\int_M \frac{d}{ds}(g(s)^{ij} D_i D_j u)\tilde{u} + \int_M (\frac{d}{ds}(R - \frac{n}{2\tau}))u\tilde{u} + \frac{1}{2}\int_M (R - \frac{n}{2\tau})|\frac{d}{ds}u|^2 dV_s \\
&= 2\int_M (-R_{pq} + \frac{1}{2\tau} g_{pq}) g^{pi}(s) g^{qj}(s) D_i D_j u\tilde{u} - \int_M g(s)^{ij} D_i D_j (\frac{d}{ds}u)\tilde{u} + \\
&+ \int_M (\frac{d}{ds}(R - \frac{n}{2\tau}))u\tilde{u} + \int_M g^{jk}(\frac{d}{dt}\Gamma_{ij}^k)\frac{\partial u}{\partial x_k}\tilde{u} + \frac{1}{2}\int_M (R - \frac{n}{2\tau})|\frac{d}{ds}u|^2 dV_s.
\end{aligned}$$

Since $\int_M g(s)^{ij} D_i D_j \frac{d}{ds} u\tilde{u} = -\int_M |\nabla_s(\frac{d}{ds}u)|^2$ and since we are on the Ricci flow, metrics $g(s)$ are uniformly bounded, after applying Cauchy-Schwartz inequality and using the uniform boundedness of the curvature operator, we get

$$\begin{aligned}
&\int_{t-A}^t \int_M |\nabla(\frac{d}{ds}u)|^2 dV_s ds + \sup_{s\in(t-A,t)} \int_M |\frac{d}{ds}u|^2 \leq \\
\leq\ & C\int_{t-A}^t \int_M |\frac{d}{ds}u|^2 dV_s ds + C\int_{t-A}^t \int_M |\nabla^2 u|^2 dV_s ds + \\
+\ & \int_M |\frac{d}{ds}u|^2 dV_s|_{s=t} + C\int_M |\nabla u|^2.
\end{aligned}$$

$\int_M |\frac{d}{ds}u(s)|^2 dV_s|_{s=t} \le C(\int_M |\Delta u_t|^2 + \int_M h(t)^2)$ where $h(s) = (\frac{n}{2\tau} - R(s))u(s)$. Since $u_t = e^{-f_t}$, where f_t are the minimizers for $\mathcal{W}$, like in the previous section we can conclude that $u_t \in W^{k,p}$, with uniform bounds on $W^{k,p}$ norms (these bounds depend on k) and therefore, $\int_M |\frac{d}{ds}u(s)dV_s|_{s=t}$ are uniformly bounded in t. This estimate together with estimates (14) and (15) gives that

$$\int_{t-A}^t \int_M |\nabla(\frac{d}{ds}u)|^2 dV_s ds \le C. \tag{17}$$

$$\sup_{s\in(t-A,t)} \int_M |\frac{d}{ds}u|^2 \le C. \tag{18}$$

If $\tilde{u} = \frac{d}{ds}u$ and $\tilde{h} = \frac{d}{ds}h$ then:

$$\frac{d}{ds}\tilde{u} = -D_s\Delta u + \tilde{h}.$$

$$\begin{aligned} D_s\Delta u = \frac{d}{ds}(g(s)^{ij}D_iD_ju) &= g(s)^{ip}g(s)^{jq}(\frac{1}{\tau}g_{pq} - 2R_{pq})D_iD_ju + g(s)^{ij}D_iD_j\tilde{u} \\ &+ g(s)^{ij}\frac{d}{ds}(\Gamma_{ij}^k)D_ku. \end{aligned}$$

$$\begin{aligned} H &= \tilde{h} - g^{ip}g^{jq}(\frac{1}{\tau}g_{pq} - 2R_{pq})D_iD_ju - g(s)^{ij}\frac{d}{ds}(\Gamma_{ij}^k)D_ku \\ &= \frac{d}{ds}\tilde{u} + \Delta\tilde{u}. \end{aligned} \tag{19}$$

All the estimates that we have got so far tell that $\int_{t-A}^t \int_M H^2$ is uniformly bounded in t. The analogous estimates to the estimates (14), (15) and (16) for u, we can get for $\frac{d}{ds}u$ (by using the evolution equation for $\frac{d}{ds}u$ and all the estimates that we have got so far by analyzing the evolution equation for u).

$$\int_{t-A}^t \int_M (|\nabla^2(\frac{d}{ds}u)|^2 dV_s ds \le C. \tag{20}$$

$$\int_{t-A}^t \int_M (\frac{d^2}{ds^2}u)^2 dV_s ds \le C. \tag{21}$$

$$\sup_{s\in(t-A,t)}\int_M |\nabla(\frac{d}{ds}u)|^2 dV_s \le C. \tag{22}$$

To obtain these estimates we have used the fact that

$$\int_M |\nabla \frac{d}{ds}u|^2 dV_{g(s)}|_{s=t} \le C(\int_M |\nabla\Delta u_t|^2 + \int_M |\nabla(R-\frac{n}{2\tau})u_t|^2,$$

where the right hand side is uniformly bounded in t, since $u_t = e^{-f_t}$ and f_t are the minimizers for $\mathcal{W}$.

By standard regularity theory, considering $\Delta u_t(s) = -\frac{d}{ds}u_t(s) + h_t(s)$ as an elliptic equation whose right hand side has uniformly bounded $W^{1,2}$ norms for $s\in(t-A,t)$ and all $t\ge A$, we have that $|u_t(s)|_{W^{3,2}} \le C$, for a uniform constant C that depends on A. Take a derivative in s of the equation $\frac{d}{ds}\tilde{u} = -\Delta\tilde{u} + H$, with $\tilde{u} = \frac{d}{ds}u$. Denote by $\bar{u} = \frac{d}{ds}\tilde{u}$. By using the estimates that we have got for $\tilde{u}$ it is easy to conclude that $\bar{u}$ satisfies the equation

$$\frac{d}{ds}\bar{u} = -\Delta\bar{u} + H_1,$$

where $H_1 = \frac{d}{ds}H + g^{ip}g^{jq}(-2R_{pq}+\frac{1}{\tau})D_iD_j\tilde{u} + g(s)^{ij}\frac{d}{ds}(\Gamma^k_{ij})D_k\tilde{u}$ and $\int_{t-A}^t\int_M H_1^2 dV_{g(s)}ds$ is uniformly bounded in t. As in the case of the previous estimates we can conclude that

$$\sup_{s\in(t-A,t)}\int_M |\frac{d}{ds}\tilde{u}|^2 dV_s \le C,$$

$$\sup_{s\in(t-A,t)}\int_M |\nabla(\frac{d}{ds}\tilde{u})|^2 dV_s \le C.$$

By regularity theory applied to the equation $\Delta\tilde{u} = -\frac{d}{ds}\tilde{u} + H$, we can get that $\frac{d}{ds}u_t(s)$ has uniformly bounded $W^{3,2}$ norms. If we go back to the parabolic equation for $u_t(s)$ we can get that $|u_t(s)|_{W^{5,2}} \le C$ for all $t\ge A$ and all $s\in(t-A,t)$. Continuing this process by taking more and more derivatives in t of our original parabolic equation we can conclude that $W^{p,2}$ norms of $u_t(s)$ are uniformly bounded for every p, by the constants that depend on A and p. Sobolev embedding theorem now gives that all $C^{k,\alpha}$ norms of $u_t(s)$

are uniformly bounded for all $t > A$ and all $s \in [t - A, t]$, by constants that depend on A and k. $\square$

Combining Theorem 16 and Lemma 14, we get that for every A there exist constants $C_k = C(k, A)$ such that $|f_t(s)|_{C^{k,\alpha}} \leq C_k$, for all $t \geq A$ and all $s \in [t - A, t]$.

3.4 Ricci soliton in the limit

In this subsection we want to finish the proof of Theorem 2.

We have uniform curvature and diameter bounds for our flow $g(t)$. We have already proved that we also have a volume noncollapsing condition along the flow, for all times $t \geq 0$. This gives a uniform lower bound on the injectivity radii. Hamilton's compactness theorem (modified to the case of our flow) gives that for every sequence $t_i \to \infty$ there exists a subsequence so that $g(t_i + t) \to h(t)$ uniformly on compact subsets of $M \times [0, \infty)$ and that $h(t)$ is a solution to the Ricci flow (1). We will show below that for each t, $h(t)$ satisfies actually a Ricci soliton equation with the Hessian of function $f_h(t)$ involved, where $f_h(t)$ is a smooth one parameter family of functions. We will now see how we get the functions $f_h(t)$, using the estimates on $f_t(s)$ from the previous subsection and Perelman's monotonicity formula.

Take any t and let f_t be a function so that $\mu(g(t), \tau) = \mathcal{W}(g(t), f_t, \tau)$. Flow f_t backward. Fix $A > 0$. Then:

$$I(t) = \mathcal{W}(g(t+A), f_{t+A}, \tau) - \mathcal{W}(g(t), f_{t+A}(t), \tau) \leq \mu(g(t+A), \tau) - \mu(g(t), \tau) \to 0 (t \to \infty).$$

$$0 \leq I(t) = \int_0^A \frac{d}{du} W(g(t+s), f_{t+A}(t+s), \tau) ds \to 0,$$

as $t \to \infty$. We will consider $u_{t_i + A}(t_i + s)$ where $s \in [0, A]$. We will divide the proof of the theorem in a few steps.

Step 16.1. $\forall A > 0$, $\lim_{i \to \infty} \frac{d}{du} W(g(s + t_i), f_{t_i + A}(s + t_i), \tau) = 0$ for almost all $s \in [0, A]$.

Proof. $I(t_i) \to 0$ by Claim 9. On the other hand

$$I(t_i) = \mathcal{W}(g(t_i+A), f_{t_i+A}, \tau) - \mathcal{W}(g(t_i), f_{t_i+A}(t_i), \tau) = \int_0^A \frac{d}{du} W(g(t_i+s), f_{t_i+A}(t_i+s), \tau) ds.$$

Since by Perelman's monotonicity formula $\frac{d}{du} W(g(t_i+s), f_{t_i+A}(t_i+s), \tau) \geq 0$, we have that $\lim_{i\to\infty} \frac{d}{du} W(g(t_i+s), f_{t_i+}(t_i+s), \tau) = 0$ for almost all $s \in [0, A]$, for

$$\int_0^A \lim_{i\to\infty} \frac{d}{du} W(g(t_i+s), f_{t_i+A}(t_i+s), \tau) ds \leq \lim_{i\to\infty} I(t_i),$$

by Fatuous lemma. □

Step 16.2. $|\tilde{u}_t(s)|_{C^{2,\alpha}} \leq C$, $\forall t$, where $\tilde{u}_t(s) = \frac{d}{ds} u_t(s)$.

Proof. Following the notation of the previous subsection, we get that:

$$\frac{d}{ds} \tilde{u}_t(s) = -\Delta \tilde{u}_t(s) + H_t(s),$$

where $H_t(s) = \frac{d}{ds} h_t(s) + g^{ip} g^{jq} (\frac{1}{\tau} g_{pq} - 2R_{pq}) D_i D_j u + g^{ij} \frac{d}{ds} (\Gamma_{ij}^k) D_k u$.

$$\tilde{u}_t(t) = \frac{d}{ds} u_t(s) = -\Delta u_t + (-\frac{n}{2\tau} + R) u_t.$$

In the previous subsection we have proved that there exist a uniform lower and an upper bound on $u_t(s)$ and that $|u_t(s)|_{W^{3,p}} \leq C(p, A)$ for all $t \geq A$ and all $s \in [t-A, t]$. Similarly we can get that $|u_t(s)|_{W^{k,p}} \leq C(k, p, A)$ and therefore $|\tilde{u}_t(s)|_{W^{k-2,p}} \leq C(k, p, A)$, $\forall t \geq A$ and all $s \in [t-A, t]$. We can get that $|\tilde{u}_t(s)|_{C^{2,\alpha}} \leq C$, for all $t \geq A$ and $\forall s \in [t-A, t]$. We can extend this to all higher order time derivatives of $u_t(s)$. □

Step 16.3. For every $A > 0$ there exists a subsequence t_i, so that the limit metric $h(s)$ of a sequence $g(t_i + s)$ is a Ricci soliton for $s \in [0, A]$.

Proof. By step 16.1 we have that

$$\lim_{i\to\infty} R_{jk}(t_i+s) + \nabla_j \nabla_k f_{t_i+A}(t_i+s) - \frac{1}{2\tau} g_{jk}(t_i+s) = 0,$$

for almost all $s \in [0,A]$ and almost all $x \in M$, since

$$\frac{d}{ds}\mathcal{W}(g(t_i+s), f_{t_i+A}(t_i+s), \tau) = (4\pi\tau)^{-\frac{n}{2}}\int_M 2\tau|R_{jk}+\nabla_j f_{t_i+A}\nabla_k f_{t_i+A}-\frac{1}{2\tau}g_{jk}|^2 dV_{g(t_i+s)}.$$

By Lemma 14 and Theorem 16, we have that $0 < C_1 \le |u_{t_i+A}(s+t_i)| \le C_2$ for all $i \ge i_0$ and all $s \in [0,A]$, for some constants C_1 and C_2 that depend on A. By step 16.2 and Theorem 16 we can find a subsequence, say $\{t_i\}$ such that $f_{t_i+A}(t_i+s)$ converges in $C^{2,\alpha}$ norm to $\tilde{f}_A(s)$ for all $s \in [0,A]$ and all $x \in M$. More precisely, for a countable dense subset $\{s_j\}$ of $[0,A]$ there exists a subsequence so that $f_{t_i+A}(t_i+s_j)$ converges in $C^{2,\alpha}$ norm to $\tilde{f}_A(s_j)$ on M. For any $s \in [0,A]$ there exists a subsequence t_{i_k} so that $f_{t_{i_k}+A}(t_{i_k}+s)$ converges to $\tilde{f}_A(s)$ in $C^{2,\alpha}$ norm. We want to show that actually $f_{t_i+A}(t_i+s) \stackrel{C^{2,\alpha}}{\to} \tilde{f}_A(s)$. For that we use the fact that $\frac{d}{ds}f_{t_i+A}(t_i+s)$ is uniformly bounded in $C^{2,\alpha}$ norm, and therefore

$$|\tilde{f}_A(s) - \tilde{f}_A(s_0)|_{C^{2,\alpha}} < \epsilon,$$

for some small $\epsilon > 0$ and some $s_0 \in \{s_j\}$ that is sufficiently close to s. We also have

$$|\tilde{f}_A(s_0) - f_{t_i+A}(t_i+s_0)|_{C^{2,\alpha}} < \epsilon,$$

for $i \ge i_0$ and

$$|f_{t_i+A}(t_i+s) - f_{t_i+A}(t_i+s_0)|_{C^{2,\alpha}} < \epsilon,$$

since $|\frac{d}{ds}f_{t_i+A}(t_i+s)|_{C^{2,\alpha}} \le C(A)$, for all $i \ge i_0$ and all $s \in [0,A]$. By triangle inequality, we now get that for every $\epsilon > 0$ there exists i_0 so that

$$|\tilde{f}_A(s) - f_{t_i+A}(t_i+s)|_{C^{2,\alpha}} < 3\epsilon,$$

for all $i \ge i_0$ and all $s \in [0,A]$.

$f_{t_i+A}(t_i+s)$ converges in $C^{2,\alpha}$ norm on M to $\tilde{f}_A(s)$, for all $s \in [0,A]$. Finally, we get that

$$R_{jk} + \nabla_j\nabla_k\tilde{f}_A(s) - \frac{1}{2\tau}h_{jk}(s) = 0, \tag{23}$$

for all $s \in [0, A]$, and for almost all $x \in M$. Because of the continuity it will hold for all $x \in M$. Since $h(s)$ is a Ricci flow, all covariant derivatives of h and the covariant derivatives of a curvature operator are uniformly bounded, and therefore $|\nabla^p \tilde{f}_A(s)| \le C(p)$, $\forall s \in [0, A]$ and all $p \ge 2$. Also we have that $|\frac{d^p}{ds^k} \nabla^p \tilde{f}_A(s)| \le C(p, k)$ where $C(p, k)$ does not depend on A, for $p \ge 2$. $\square$

Step 16.4. We can glue all the functions $\tilde{f}_A$ that we get for different values of A, to get a function $f_h(s)$ defined on $M \times [0, \infty)$, which defines our metric $h(s)$ as a soliton type solution for all times $s \ge 0$.

Proof. Take any increasing sequence $A_j \to \infty$. For every A_j, by the previous step we can extract a subsequence t_i so that $f_{t_i + A_j}(t_i + s) \stackrel{C^{2,\alpha}}{\to} \tilde{f}_{A_j}(s)$ for all $s \in [0, A_j]$. Diagonalization procedure gives a subsequence so that $f_{t_i + A_j}(s) \stackrel{C^{2,\alpha}}{\to} \tilde{f}_{A_j}(s)$ for all j and all $s \in [0, A_j]$. For this subsequence t_i we have that $g(t_i + t) \to h(t)$, uniformly on compact subsets of $\times[0, \infty)$. Compare the functions $\tilde{f}_{A_j}$ and $\tilde{f}_{A_k}$ for $j < k$, on the interval $[0, A_j]$. We know that they both satisfy

$$\Delta_{h(s)} \tilde{f}_{A_r} + R(h(s)) - \frac{n}{2\tau} = 0,$$

and therefore $\Delta_{h(s)}(\tilde{f}_{A_j} - \tilde{f}_{A_k}) = 0$. Since M is compact, this implies that $\tilde{f}_{A_k}(s) = \tilde{f}_{A_j}(s) + c_{A_j}^{A_k}(s)$, for $s \in [0, A_j]$, where $c_{A_j}^{A_k}(s)$ is a constant function for every $s \in [0, A_j]$. On the other hand, because of the integral normalization condition, we have

$$(4\pi\tau)^{-\frac{n}{2}} \int_M e^{-\tilde{f}_{A_j}(s)} dV_{h(s)} = 1,$$

$$(4\pi\tau)^{-\frac{n}{2}} \int_M e^{-\tilde{f}_{A_k}(s)} dV_{h(s)} = 1 = e^{-c_{A_j}^{A_k}(s)} (4\pi\tau)^{-\frac{n}{2}} \int_M e^{-\tilde{f}_{A_j}(s)} dV_{h(s)},$$

which implies that $c_{A_j}^{A_k}(s) = 0$ for all $s \in [0, A_j]$ and all $k \ge j$. Therefore $\tilde{f}_{A_j}(s) = \tilde{f}_{A_k}(s)$ for all $s \in [0, A_j]$. Define a function $f_h(s)$ in the following

way. Let $f_h(s) = \tilde{f}_{A_j}(s)$, for all $s \in [0, A_j]$ and all $A_j \to \infty$. $f_h(s)$ is a well defined function because of the previous discussion. We also have that

$$R(h(s))_{pq} + \nabla_p \nabla_q f_h(s) - \frac{1}{2\tau} h(s)_{pq} = 0, \tag{24}$$

holds for all $s \in [0, \infty)$. The definition of $f_h(s)$ does not depend on a choice of an increasing sequence A_j. Namely, if B_j were another increasing sequence and if $f_{h'}(s)$ were functions defined using the sequences B_j and t_i (t_i is the same sequence as above), then at each time both functions $f_h(s)$ and $f_{h'}(s)$ would satisfy the same equation (24) and the same integral normalization condition. Therefore $f_h(s) = f_{h'}(s)$ for all $s \in [0, \infty)$. □

3.5 Some properties of the limit solitons

Let t_i be any sequence converging to infinity. Then as we have seen earlier, there exists a subsequence such that $g(t_i + s) \to h(s)$, where $h(s)$ is a Ricci soliton. Let $\hat{R}(h(t)) = \min R(h(t))$. We will first state a theorem that R. Hamilton proved in his paper [9].

Theorem 17 (Hamiton). *Under the normalized Ricci flow, whenever $\hat{R} \leq 0$, it is increasing, whereas if ever $\hat{R} \geq 0$ it remains so forever.*

We will use the proof of Theorem 17 to prove the following lemma.

Lemma 18. *Under the assumptions of Theorem 2, $\hat{R}(h(t)) \geq 0$, $\forall t$, for the limit metric $h(t)$ of any sequence of metrics $g(t_i)$, where $g(t)$ is a solution of*

$$\frac{d}{dt} g_{jk} = -2R_{jk}(g(t)) + \frac{1}{\tau} g_{jk}(t).$$

Proof. Assume that there exists t_0 such that $\hat{R}(h(t_0)) < 0$. Without loss of generality assume that $t_0 = 0$. Since $g(t_i) \to h(0)$ as $i \to \infty$, there exists i_0, so that for all $i \geq i_0$ $\hat{R}(g(t_i)) < 0$. The evolution equation for R is

$$\frac{d}{dt} R = \Delta R + 2|\mathrm{Ric}|^2 + \frac{2}{n} R(R - \frac{n}{2\tau}).$$

This implies

$$\frac{d}{dt}\hat{R} \geq \frac{2}{n}\hat{R}(\hat{R} - \frac{n}{2\tau}).$$

If $\hat{R} \leq 0$, then $\hat{R}$ is increasing (since $\frac{d}{dt}\hat{R} \geq 0$). If $\hat{R} \geq 0$ at some time it can not go negative at later times. If there existed $t > t_{i_0}$ such that $\hat{R}(g(t)) \geq 0$, then $\hat{R} \geq 0$ would remain so forever, for all $s \geq t$ and therefore we could not have $\hat{R}(g(t_i)) < 0$ for $t_i > t$. That contradicts the fact that $\hat{R}(g(t_i)) < 0$ for all $i \geq i_0$. Therefore $\forall t \geq t_{i_0}$ we have that $\hat{R}(g(t)) < 0$.

$$\frac{d\hat{R}}{dt} \geq \frac{2}{n}\hat{R}(\hat{R} - \frac{n}{2\tau}) \geq 0,$$

for all t big enough. That implies $\hat{R}$ is increasing and therefore there exists $\lim_{t\to\infty}\hat{R}(g(t)) = -C \leq 0$. Moreover $\hat{R}(h(s)) = -C$ for all s. Since $\lim_{i\to\infty}\hat{R}(g(t_i)) = \hat{R}(h(0)) < 0$, $C > 0$. We also have that

$$\frac{d\hat{R}(h(s))}{ds} \geq -\frac{2}{n}\hat{R}(h(s))(\frac{n}{2\tau} - \hat{R}(h(s))) = \frac{2}{n}C(\frac{n}{2\tau} + C) \geq 0.$$

The left hand side of the above inequality is zero and therefore we get that $C = -\frac{n}{2\tau}$ or $C = 0$. Since $C > 0$, we get a contradiction. Therefore $R(h(t)) \geq 0$ for all t, what we wanted to prove. □

Remark 19. *Let (M, g) be a compact manifold and $g(t)$ be a Ricci flow on M. Since*

$$\frac{d}{dt}\mathcal{W} = \int_M 2\tau|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij}|^2(4\pi\tau)^{-\frac{n}{2}}e^{-f}dV,$$

$\mathcal{W}(g, f, \tau) = const$ along the flow, if g is a Ricci soliton satisfying the equation

$$R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau}g_{ij} = 0.$$

Let $t_i \to \infty$ and $s_i \to \infty$ be two sequences such that $g(t_i + t) \to h(t)$ and $g(s_i + t) \to h'(t)$ where $h(t)$ and $h'(t)$ are 2 Ricci solitons on M that have been constructed earlier. We have proved that

$$R_{jk}(h) + \nabla_j\nabla_k f_h(t) - \frac{1}{2\tau}h_{jk} = 0,$$

$$R_{jk}(h') + \nabla_j\nabla_k f_{h'}(t) - \frac{1}{2\tau}h'_{jk} = 0,$$

where

$$f_h(t) = \lim_{j\to\infty}\lim_{i\to\infty} f_{A_j+t_i}(t_i+t),$$
$$f_{h'}(t) = \lim_{j\to\infty}\lim_{i\to\infty} f_{B_j+s_i}(s_i+t),$$

for some increasing sequences $A_j \to \infty$ and $B_j \to \infty$. By Remark 19 we know that $\mathcal{W}(h(t), f_h(t), \tau) = C_1$ and $W(h'(t), f_{h'}(t), \tau) = C_2$ are constant along the flows $h(t)$ and $h'(t)$ respectively.

Lemma 20. *$C_1 = C_2$, i.e. $\mathcal{W}(h(t), f_h(t), \tau)$ is a same constant for all solitons $h(t)$ that arise as limits of sequences of metrics of our original flow $g(t)$ (1) on a compact manifold M.*

Proof.

$$\begin{aligned}
& \mathcal{W}(g(t_i+t), f_{t_i+A_j}(t_i+t), \tau) - \mathcal{W}(g(s_i), f_{s_i+B_j}(s_i), \tau) \le \\
\le\ & \mathcal{W}(g(t_i+A_j), f_{t_i+A_j}(t_i+A_j), \tau) - \mathcal{W}(g(s_i), f_{s_i}(s_i), \tau) = \\
=\ & \mu(g(t_i+A_j), \tau) - \mu(g(s_i), \tau) \to 0, \qquad (25)
\end{aligned}$$

where we have used the fact that $\mathcal{W}(g(t), f(t), \tau)$ increases in t along the flow (1) and the fact that $f_{s_i}(s_i) = f_{s_i}$ is a minimizer for $\mathcal{W}(g(s_i), f, \tau)$ over all f belonging to a set $\{f \mid \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV_{g(s_i)}\}$. Similarly,

$$\begin{aligned}
& \mathcal{W}(g(t_i+t), f_{t_i+A_j}(t_i+t), \tau) - \mathcal{W}(g(s_i), f_{s_i+B_j}(s_i), \tau) \ge \\
\ge\ & \mathcal{W}(g(t_i+t), f_{t_i+t}(t_i+t), \tau) - \mathcal{W}(g(s_i+B_j), f_{s_i+B_j}(s_i+B_j), \tau) = \\
=\ & \mu(g(t_i+t), \tau) - \mu(g(s_i+B_j), \tau) \to 0, \qquad (26)
\end{aligned}$$

when $i \to \infty$. From equations (25) and (26), letting $i \to \infty$ we get

$$\mathcal{W}(h(t), \tilde{f}_{A_j}(t), \tau) - \mathcal{W}(h'(0), \tilde{f}'_{B_j}(0), \tau) \le 0.$$
$$W(h(t), \tilde{f}_{A_j}(t), \tau) - W(h'(0), \tilde{f}'_{B_j}(0), \tau) \ge 0.$$

Let $j \to \infty$ to get

$$C_1 = \mathcal{W}(h(t), f_h(t), \tau) = \mathcal{W}(h'(0), f_{h'}(0), \tau) = C_2.$$

$\square$

Lemma 21. *For every Ricci soliton $h(t)$ that arises as a limit of some sequence of metrics of our original flow $g(t)$, the corresponding function $f_h(t)$, that we have constructed before, is a minimizer for Perelman's functional $\mathcal{W}$ with respect to a metric $h(t)$.*

Proof. We will first proof the following claim.

Claim 22. *There exists a sequence $t_i \to \infty$ such that $g(t_i + t) \to h(t)$ as $i \to \infty$, where $h(t)$ is a Ricci soliton satisfying $R_{jk}(h) + \nabla_j \nabla_k f_h - \frac{1}{2\tau} h_{jk} = 0$ and $f_h(t)$ is a minimizer for $\mathcal{W}(h(t), f, \tau)$.*

Proof of the Claim. Let $H(t) = (4\pi\tau)^{-n/2} \int_M 2\tau |R_i j + \nabla_i \nabla_j f_t - \frac{1}{2\tau} g_{ij}|^2 dt$, where f_t is a function such that $\mu(g(t), \tau) = W(g(t), f_t, \tau)$. If we flow f_t backward by the equation

$$\frac{d}{dt} f = -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau},$$

starting at time t, for every $t > 0$ we get solutions $f_t(s)$. Look at $F_t(s) = \mathcal{W}(g(s), f_t(s), \tau)$. We know that

$$\frac{d}{ds} F_t(s) = (4\pi\tau)^{-\frac{n}{2}} \int_M 2\tau |R_{jk} + \nabla_j \nabla_k f_t(s) - \frac{1}{2\tau} g(s)_{jk}|^2 dV_{g(s)}.$$

$F_t(s)$ is a continuous function in $s \in [0, t]$ and $\lim_{s \to t} \frac{d}{ds} F_t(s) = H(t)$. Therefore there exists a left derivative of $F_t(s)$ at point t and $(F_t)'_-(t) = H(t)$ for every $t > 0$. Moreover, $g(t)$ and all the derivatives of f_t up to the second order are Lipshitz functions in t (this follows from the estimates in the previous subsections) and therefore

$$\mu(t) := \mu(g(t),\tau) = \inf_{\{f \mid \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f}=1\}} \mathcal{W}(g(t),f,\tau)$$

is a Lipshitz function in t as well, i.e. $k(t) = F_t(t) = \mathcal{W}(g(t), f_t, \tau)$ is a Lipshitz function in t. This tells that $k(t)$ is differentiable in t, almost everywhere. Our discussion then implies that $k'(t) = H(t)$ in a sense of distributions.

$$\begin{aligned}
\int_\delta^\infty H(t)dt &= \lim_{K\to\infty}\int_\delta^K k'(t)dt \\
&= \lim_{K\to\infty} W(g(K), f_K, \tau) - W(g(\delta), f_\delta, \tau) \\
&= \lim_{K\to\infty} (\mu(g(K),\tau) - \mu(g(\delta),\tau) \le C, \qquad (27)
\end{aligned}$$

where $\delta > 0$ and C is some uniform constant. We have that $\int_\delta^\infty H(t) \le C$. This implies that there exists a sequence $t_i \to \infty$ such that $H(t_i) \to 0$ as $i \to \infty$, i.e.

$$\lim_{i\to\infty} (R_{jk} + \nabla_j\nabla_k f_{t_i} - \frac{1}{2\tau} g_{jk})(t_i) = 0.$$

By what we have proved before, after extracting a subsequence we can assume that $g(t_i) \to h(0)$ smoothly and $f_{t_i} \to \tilde{f}$ in $C^{2,\alpha}$ norm, where by Theorem 12 $\tilde{f}$ is a minimizer for $\mathcal{W}$ with respect to metric $h(0)$. Therefore,

$$R_{jk}(h(0)) + \nabla_j\nabla_k\tilde{f} - \frac{1}{2\tau}h_{jk}(0) = 0. \qquad (28)$$

On the other hand $g(t_i + t) \to h(t)$ as $i \to \infty$ where $h(t)$ is a Ricci soliton and

$$R_{jk}(h(t)) + \nabla_j\nabla_k f_h(t) - \frac{1}{2\tau}h_{jk}(t) = 0, \qquad (29)$$

where $f_h(t) = \lim_{j\to\infty}\lim_{i\to\infty} f_{t_i+A_j}(t_i + t)$, for some sequence $A_j \to \infty$. From equations (28) and (29) we have that $\Delta(f_h(0) - \tilde{f}) = 0$, i.e. $f_h(0) = \tilde{f} + C$ for some constant C. We know that $\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-\tilde{f}} dV_{h(0)} = 1$, since $\tilde{f}$ is a minimizer. From the construction of $f_h(t)$ it follows that

$\int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f_h(0)} dV_{h(0)} = 1$ and therefore $\tilde{f} = f_h(0)$. Since there exists a finite limit, $\lim_{t\to\infty} \mu(g(t), \tau)$, we have that $\mu(h(0), \tau) = \mu(h(t), \tau)$ for all t. This implies that

$$\begin{aligned}\mu(h(t), \tau) &= \mu(h(0), \tau) = \mathcal{W}(h(0), \tilde{f}, \tau) \\ &= \mathcal{W}(h(0), f_h(0), \tau) = \mathcal{W}(h(t), f_h(t), \tau),\end{aligned}$$

where we have used the fact that $\mathcal{W}$ is constant along a soliton. This means that $f_h(t)$ is a minimizer for $\mathcal{W}$ with respect to a metric $h(t)$, for every $t \geq 0$. □

To continue the proof of Lemma 21 take any sequence $s_i \to \infty$. By a sequential convergence of our original flow $g(t)$ to Ricci solitons, after extracting a subsequence we may assume that $g(s_i + t) \to h'(t)$ as $i \to \infty$ where $h'(t)$ is a Ricci soliton. Take a soliton $h(t)$ with the properties as in Claim 22. From the convergence of $\mu(g(t), \tau)$ we know that $\mu(h'(t), \tau) = \mu(h(s), \tau)$ for all t and all s.

$$\mu(h'(t), \tau) = \mu(h(s), \tau) = \mathcal{W}(h(s), f_h(s), \tau). \tag{30}$$

By Lemma 20 we have that $\mathcal{W}(h(s), f_h(s), \tau) = \mathcal{W}(h'(t), f_{h'}(t), \tau)$ for all s and t. Combining this with (30) gives that $\mu(h'(t), \tau) = \mathcal{W}(h'(t), f_{h'}(t), \tau)$, i.e. $f_{h'}(t)$ is a minimizer for $h'(t)$ for every t. □

One useful property of the sequential soliton limits of our flow (1) is that all limit solitons are the solutions of the normalized flow equation

$$\frac{d}{dt} h_{ij} = -2R_{ij} + \frac{2}{n} r(h(t)) h_{ij},$$

where $r(h(t)) = \frac{1}{\mathrm{Vol}_{h(t)} M} \int_M R(h(t)) dV_{h(t)}$. In the case of any of our soliton limits, we have that $R(h(t)) + \Delta f_h(t) - \frac{n}{2\tau} = 0$ and therefore $r = r(h(t)) = \frac{n}{2\tau}$ for all $t \geq 0$.

Remark 23. *Let $t_i \to \infty$ and $g(t_i + t) \to h(t)$, where $h(t)$ is an Einstein metric with an Einstein constant $\frac{1}{2\tau}$. If $\mathrm{Vol}_{h'}(M) = \mathrm{Vol}_h(M)$, for any other*

limit soliton h', then h' is an Einstein metric with the same Einstein constant $\frac{1}{2\tau}$.

Proof. The fact that h is Einstein metric implies that $\nabla_i\nabla_j f_h = -2R_{ij} + \frac{1}{\tau}h_{ij} = 0$, that is $\Delta f_h = 0$. Since M is compact, $f_h = C$ such that $(4\pi\tau)^{-n/2}e^{-C}\mathrm{Vol}_h(M) = 1$. An easy computation shows that $\mu(h,\tau) = \mathcal{W}(h,C,\tau) = C - \frac{n}{2}$, and therefore $\mu(h',\tau) = \mu(h,\tau) = C - \frac{n}{2}$. Then, $(4\pi\tau)^{-n/2}e^{-C}\mathrm{Vol}_{h'}(M) = 1$, implies that $f = C$ is a minimizer for $\mathcal{W}$ with respect to h' as well. This yields

$$\tau(2\Delta f - |\nabla f|^2 + R(h')) + f - n = C - \frac{n}{2},$$

that is

$$R(h') = \frac{n}{2\tau}.$$

From

$$\Delta f_{h'} = \frac{n}{2\tau} - R(h') = 0,$$

we get that $f_{h'} = C$ and therefore

$$R_{ij}(h') + \nabla_i\nabla_j f_{h'} - \frac{1}{2\tau}h'_{ij} = 0,$$

yields $R_{ij}(h') = \frac{1}{2\tau}h'_{ij}$. □

In the discussion that follows we will use Moser's weak maximum principle. We will state it below, for a reader's convenience.

Lemma 24 (Moser's weak maximum principle). *Let $g = g(t)$, $0 \le t < T$, be a smooth family of metrics, b a nonnegative constant and f a nonnegative function on $M \times [0,T)$ which satisfies the partial differential inequality*

$$\frac{df}{dt} \le \Delta f + bf,$$

on $M \times [0,T]$, where Δ refers to a Laplacian at time t. Then for any $x \in M$, $t \in [0,T)$,

$$|f(x,t)| \le c\frac{1}{\sqrt{V}}e^{cHd}\max(1,d^{\frac{n}{2}})(b+l+\frac{1}{t})^{\frac{1+n/2}{2}}e^{cbt}||f_0||_{L^2},$$

where c is a positive constant depending only on n and $d = \max_{0\le t\le T}\mathrm{diam}(M,g(t))$, $H = \max_{0\le t\le T}\sqrt{||\mathrm{Ric}||_{C^0}}$, $f_0 = f(\cdot,0)$, $V = \min_{0\le t\le T}\mathrm{Vol}_{g(t)}(M)$.

The following remark will give us a condition that will imply obtaining the Einstein metrics in the limit.

Remark 25. *If $g(t)$ is a solution to $(g_{ij})_t = -2R_{ij} + \frac{1}{\tau}g_{ij}$, for $t \in [0,\infty)$ such that*

1. *A curvature operator and a diameter are uniformly bounded along the flow.*
2. *$0 \le R(x,t) \le \frac{n}{2\tau}$ for all $x \in M$ and all $t \in [0,\infty)$.*

Then all the solitons that arise as limits of the subsequences of our flow $g(t)$ are Einstein metrics with scalar curvatures $R = \frac{n}{2\tau}$ and $T_{ij}(t)$ converge to zero, uniformly on M as $t \to \infty$. $T_{ij} = R_{ij} - \frac{R}{n}g_{ij}$ is a traceless part of the Ricci curvature.

Proof of the Remark. Notice that now we do not make an assumption that one of the metrics that we get in a limit is an Einstein metric. Look at the evolution equation for $r(t) = \frac{1}{\mathrm{Vol}_t(M)}\int_M R\,dV_t$,

$$\frac{d}{dt}r(t) = \frac{1}{\mathrm{Vol}_t(M)}(2\int_M |T|^2 + (1-\frac{2}{n})\int_M R(\frac{n}{2\tau} - R) + r(r - \frac{n}{2\tau}).$$

$R \le \frac{n}{2\tau}$ implies $r(t) \le \frac{n}{2\tau}$ and therefore

$$\frac{d}{dt}r(t) \ge \frac{2}{\mathrm{Vol}_t(M)}\int_M |T|^2 + r(r - \frac{n}{2\tau}). \tag{31}$$

We have proved that in the case of flow $g(t)$, a volume noncollapsing condition holds for all times $t \ge 0$. $\frac{d}{dt}\ln(\mathrm{Vol}_t(M)) = \frac{n}{2\tau} - r$ and $C_1 \le \mathrm{Vol}_t(M) \le C_2$ give that $\int_0^\infty(\frac{n}{2\tau} - r(t))dt < \infty$. We can integrate the inequality (31) in

$t \in [0, \infty)$. This, together with the uniform estimates on $\mathrm{Vol}_t(M)$ and $r(t)$ give that

$$\int_0^\infty \int_M |T|^2 dV_t \le C. \tag{32}$$

Following the calculations in Hamilton's paper [6], Rugang computed the evolution equation for T under a normalized Ricci flow ([12]). In the case of flow (1) we have

$$\frac{d}{dt}|T|^2 = \Delta |T|^2 - 2|\nabla T|^2 + 4\mathrm{Rm}(T)\cdot T + \frac{4}{n}(R - \frac{n}{2\tau})|T|^2. \tag{33}$$

Since the curvature operators of $g(t)$ are uniformly bounded, we derive from equation (33) that

$$\frac{d}{dt}|T| \le \Delta |T| + C|T|.$$

Applying Lemma 24 to this differential inequality and intervals $[t-1, t+1]$ where $t > 1$, we derive

$$|T|^2(x,t) \le ||T||^2(t)_{C^0(M)} \le C(\int_{M_{t-1}} |T|^2),$$

where $M_t = (M, g(t))$. Integrate this inequality in $t \in [k, k+1]$, for all $k \ge k_0$ and sum up all the inequalities that we get this way. We get

$$\int_{k_0}^\infty ||T||^2 dt \quad \le \quad C\sum_{k \ge k_0} \int_k^{k+1} (\int_{M_{t-1}} |T|^2) dt$$

$$\int_{k_0}^\infty ||T||^2 dt \le C\int_{k_0}^\infty \int_M |T|^2 dV_{t-1} dt, \tag{34}$$

where dV_{t-1} is a volume form for metric $g(t-1)$. $\int_M |T|^2 dV_{t-1} \le C\int_M |T|^2 dV_t$, because $\frac{d}{dt}\ln \mathrm{Vol}_t = \frac{n}{2\tau} - R$ and the curvatures of $g(t)$ are uniformly bounded. The right hand side of inequality (34) is bounded by a uniform constant, because of the estimate (32). Therefore $\int_{k_0}^\infty ||T||^2 dV_t \le C$.

If there exists (p, t_0) such that $|T|^2(p, t_0) > \epsilon$, then there is a small neighbourhood of (p, t_0) in $M \times [0, \infty)$, say $U_\delta(p, t_0) = B_p(\delta, t_0) \times [t_0 - \delta, t_0 + \delta]$ such that $|T|^2(x,t) \ge \frac{\epsilon}{2}$ for all $(x,t) \in U_\delta(p, t_0)$. This follows from the fact that in the case of a Ricci flow, a bound $|\mathrm{Rm}| \le C$ implies $|D^k D_t^l \mathrm{Rm}| \le$

$C(k,l)$. Costant δ does not depend on a point $(p, t_0) \in M \times [0, \infty)$, since all our bounds and estimates are uniform.

If there existed $\epsilon > 0$ and a sequence of points $(p_i, t_i) \in M \times [0, \infty)$, with $t_i \to \infty$ such that $|T(p_i, t_i)| \geq \epsilon$ then we would have that $||T||_{C^0} \geq \frac{\epsilon}{2}$ for all $t \in [t_i - \delta, t_i + \delta]$ and for all i. This would imply $C \geq \int_0^\infty ||T||^2 dV_t \geq \sum_{i=0}^\infty \epsilon\delta = \infty$. This is impossible. Therefore, $||T||_{C^0(M_t)} \to 0$ as $t \to \infty$.

$\frac{d}{dt} \ln(\mathrm{Vol}_t) = \frac{n}{2\tau} - R \geq 0$ for all t imply that there exists a finite $\lim_{t\to\infty} \mathrm{Vol}_t$ for every $x \in M$ (otherwise we can argue as in the previous paragraph). If we integrate this equation in $t \in [0, \infty)$, we will get that $\int_0^\infty (\frac{n}{2\tau} - R) dt < \infty$. As in the case for a traceless part of the Ricci curvature T, we can conclude that $\lim_{t\to\infty} R = \frac{n}{2\tau}$ uniformly on M.

We can conclude that under the assumptions given at the beginning of this remark, for every sequence $t_i \to \infty$ we can find a subsequence such that $g(t_i + t) \to h(t)$, where $h(t)$ is an Einstein soliton with scalar curvature $\frac{n}{2\tau}$. We also know that $R_{ij} - \frac{1}{2\tau} g_{ij} \to 0$ as $t \to \infty$, uniformly on M and that there exists $\lim_{t\to\infty} \mathrm{Vol}_t$. $\square$

To conclude, we have proved a sequential convergence of a solution of a τ-flow towards solitons (generalizations of Einstein metrics), under uniform curvature and diameter assumptions. We still do not know whether we get a unique soliton (up to diffeomorphisms) in the limit or not. All observations in this subsection are in favour of the uniqueness of a soliton in the limit.

References

[1] H.D.Cao: Deformation of Kahler metrics to Kahler-Einstein metrics on comapct Kahler manifolds; Invent. math. 81 (1985) 359–372.

[2] J. Cheeger, T. Colding: On the structure of spaces with Ricci curvature bounded below.

[3] T. Colding: Ricci curvature and volume convergence; Annals of Mathematics, 145 (1997), 477–501.

[4] D. Deturck: Deforming metrics in the direction of their ricci tensors; J. Diff. Geom. 18 (1983), 157–162.

[5] D. Glickenstein: Precompactness of solutions to the Ricci flow in the absence of injectivity radius estimates; preprint arXiv:math.DG/0211191 v2.

[6] R. Hamilton: Three-manifolds with positive Ricci curvature, Journal of Differential Geometry 17 (1982) 225–306.

[7] A compactness property for solutions of the Ricci flow, Amer.J.Math. 117 (1995) 545–572.

[8] The formation of singularities in the Ricci flow, Surveys in Differential Geometry, vol. 2, International Press, Cambridge, MA (1995) 7–136.

[9] R. Hamilton:Non-singular solutions of the Ricci flow on 3 manifolds, Communications in analysis and geometry vol. 7 (1999) 695–729.

[10] G. Perelman: The entropy formula for the Ricci flow and its geometric applications, preprint.

[11] Rothaus: Logarithmic Sobolev inequality and spectrum, Journal of Dif. Anal. 42 (1981) 109–120.

[12] Rugang Ye: Ricci flow, Einstein metrics and space forms, Transactions of the american mathematical society, volume 338, number 2 (1993) 871–895.